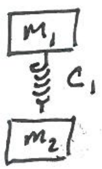
An interesting thing happens when you solve for displacement functions in a spring system with no fixed parts:

EX: Find the fundamental modes of oscillation



$m_1 = 1$

$m_2 = 1/2$

$c_1 = 3$

stiffness matrix.

$$K = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$M^{-1}K = \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix}$$

eigenvalues.

$$\det \begin{bmatrix} 3-\lambda & -3 \\ -6 & 6-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 9\lambda + 0 = 0$$

$$\lambda(\lambda - 9) = 0$$

$$\lambda = 0, 9$$

($M^{-1}K$ has 0 as an eigenvalue!)

eigenvectors.

$\lambda = 0$, $\omega = 0$:
$$\begin{bmatrix} 3-0 & -3 \\ -6 & 6-0 \end{bmatrix} = L \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$$
↳ eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 9$, $\omega = 3$:
$$\begin{bmatrix} 3-9 & -3 \\ -6 & 6-9 \end{bmatrix} = L \begin{bmatrix} -6 & -3 \\ 0 & 0 \end{bmatrix}$$
↳ eigenvector $= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

fund. modes

$\omega = 0$:
$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{\cos(0\cdot t)}^{\,1} + \cancel{d_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(0\cdot t)}^{\,0}$$
$$= \boxed{c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad ??$$

$\omega = 3$:
$$\boxed{c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cos(3t) + d_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sin(3t)}$$

The "fundamental mode" for $\omega = 0$ does not describe oscillation at all!!!

→ In fact, it is incomplete... since no part of the system is held still, the entire system can move up or down

$$\boxed{c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t}$$

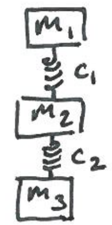
← Correct first fundamental mode

Every spring system with no connection outside will have $M^{-1}K$ with eigenvalue $\lambda = 0$

and eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$ all 1s

corresponding to moving the entire system up or down.

EX: Find fundamental modes of oscillation



$m_1 = 3/4$ $\quad c_1 = 3$

$m_2 = 1$ $\quad c_2 = 2$

$m_3 = 1/2$

stiffness matrix.
$$K = \begin{bmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$M^{-1}K = \begin{bmatrix} 4 & -4 & 0 \\ -3 & 5 & -2 \\ 0 & -4 & 4 \end{bmatrix}$$

eigenvalues
$$\det \begin{bmatrix} \boxed{\begin{matrix} 4-\lambda \\ -3 \\ 0 \end{matrix}} & \begin{matrix} -4 \\ 5-\lambda \\ -4 \end{matrix} & \begin{matrix} 0 \\ -2 \\ 4-\lambda \end{matrix} \end{bmatrix} = 0$$

expand down column #1

$$(4-\lambda) \det \begin{bmatrix} 5-\lambda & -2 \\ -4 & 4-\lambda \end{bmatrix} + 3 \det \begin{bmatrix} -4 & 0 \\ -4 & 4-\lambda \end{bmatrix} = 0$$

$$\underline{\underline{(4-\lambda)}} (\lambda^2 - 9\lambda + 12) + 3(-4)\underline{\underline{(4-\lambda)}} = 0$$

$$(4-\lambda)(\lambda^2 - 9\lambda + 12 - 12) = 0$$

$$(4-\lambda)(\lambda - 9)\lambda = 0$$

$$\lambda = 0, 4, 9 \qquad (\omega = 0, 2, 3)$$

eigenvectors

$\lambda = 0$, $\omega = 0$
$$\begin{bmatrix} 4 & -4 & 0 \\ -3 & 5 & -2 \\ 0 & -4 & 4 \end{bmatrix} = L \begin{bmatrix} 4 & -4 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvect $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 4$, $\omega = 2$
$$\begin{bmatrix} 0 & -4 & 0 \\ -3 & 1 & -2 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \to b = 0 \\ \to -3a + b - 2c = 0 \\ -3a - 2c = 0 \end{matrix}$$

eigenvect $= \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

$\lambda = 9$, $\omega = 3$
$$\begin{bmatrix} -5 & -4 & 0 \\ -3 & -4 & -2 \\ 0 & -4 & -5 \end{bmatrix} = L \begin{bmatrix} -5 & -4 & 0 \\ 0 & -8/5 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 4 \\ -5 \\ 4 \end{bmatrix}$$

fundamental modes.

$(\lambda=0)$ $\omega = 0$ $\quad c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$

$(\lambda=4)$ $\omega = 2$ $\quad c_2 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \cos(2t) + d_2 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \sin(2t)$

$(\lambda=9)$ $\omega = 3$ $\quad c_3 \begin{bmatrix} 4 \\ -5 \\ 4 \end{bmatrix} \cos(3t) + d_3 \begin{bmatrix} 4 \\ -5 \\ 4 \end{bmatrix} \sin(3t)$

Now for some MatLab code:

Recall: Matrices are entered into MatLab using square brackets with semi-colon separating rows

```
>> K = [ 4 -4 0; -3 5 -2; 0 -4 4]
```

The function eig(...) computes eigenvalues

```
>> eig(K)
```

⤳ MatLab will return $\begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix}$

To get eigenvalues and eigenvectors you must catch the return value correctly

```
>> [V, D] = eig(K)
```

⤳ V = matrix whose columns are the eigenvectors of K (normalized so that length is 1)

D = matrix whose diagonal entries are eigenvalues

To get individual eigenvectors, you need single columns of V

```
>> v1 = V(:, 1)
>> v2 = V(:, 2)
>> v3 = V(:, 3)
```

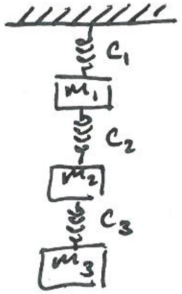
← elements of V from [all rows (:) column 1]

The eigenvalues are diagonal elements of D

```
>> lambda1 = D(1, 1)
>> lambda2 = D(2, 2)
>> lambda3 = D(3, 3)
```

← element of D from [row 1 column 1]

EX: Use MatLab to solve for displacement of masses and plot positions.



$c_1 = 4 \qquad m_1 = 1/4$

$c_2 = 3 \qquad m_2 = 1/3 \qquad$ if $\qquad u(0) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad u'(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$c_3 = 4 \qquad m_3 = 1$

stiffness matrix.

```
>> K = [7 -3 0; -3 7 -4; 0 -4 4]
```

1/mass matrix.

```
>> Minv = diag([1/(1/4) 1/(1/3) 1/(1)])
```

eigenvalues.

```
>> [V, D] = eig(Minv * K)
```

frequency of fundamental modes.

```
>> freq = sqrt(diag(D))
```

coefficients. $\left(Vc = u(0) \quad \& \quad V(\omega d) = u'(0) \right)$

```
>> c = V \ [-1; 1; 0]
>> d = (V \ [1; 0; 1]) ./ freq
```

(Recall: $Vc = u(0) \rightsquigarrow c = V \backslash u(0)$

Division by matrix on the left is written in Matlab as A \ b)

displacements of masses.

```
>> syms t
```

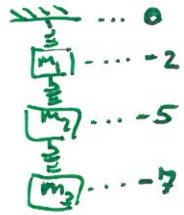
Set t to be a "symbolic variable"

```
>> u = V * (c.*cos(freq*t) + ...
            d.*sin(freq*t))
```

only needed if you cannot fit on one line

plot positions.

Assume equilibrium positions are



→ Remember: u = displacement not position

```
>> hold on; range = [0, 2*pi, -8, 0]
>> ezplot(-2+u(1), range)
```

```
>> ezplot(-5 + u(2), range)
>> ezplot(-7 + u(3), range)
```

"Animations" in MatLab

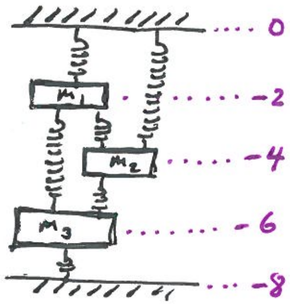
Masses oscillating over time are more fun to represent with animations, changing over time, instead of boring graphs.

EX: Suppose a mass system oscillates with eigenvectors and eigenvalues

$$\lambda_1 = 1 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 4 \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\lambda_3 = 9 \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$



Use Matlab to animate fundamental modes and mixed oscillations.

first fundamental mode.

```
>> center = [-2; -4; -6]
>> v1 = [1; 1; -1]
>> for t = 0:.1:6*pi
      pos = v1 * cos(t) + center;
      scatter([0;0;0;0;0], [0; pos; -8])
      pause(.05)
   end
```

second fundamental mode.

```
>> center = [-2; -4; -6]
>> v2 = [1; 0; -2]
>> for t = 0:.1:6*pi
      pos = v2 * cos(2*t) + center;
      scatter([0;0;0;0;0], [0; pos; -8])
      pause(.05)
   end
```

mixed oscillation. $\left(c = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/2 \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$

```
>> center = [-2; -4; -6]
>> v1 = [1; 1; -1]
>> v2 = [1; 0; -2]
>> v3 = [1; -1; 1]
>> for t = 0: .1 : 6*pi
     pos = 1/2 * v1 * cos(t) + ...
           1/3.* v2 * cos(2*t) + ...
           1/2 * v3 * cos(3*t) + ...
           center;
     scatter([0;0;0;0;0], [0; pos; -8])
     pause (.05)
   end
```

By changing our loop slightly, we can leave behind traces of the paths of masses as they oscillate.

(Additions marked in green)

first fundamental mode.

```
>> center = [-2; -4; -6]; v1 = [1; 1; 1]
>> hold on
>> for t = 0: .1 : 6*pi
     pos = v1 * cos(t) + center;
     scatter([t;t;t;t;t], [0; pos; -8])
     pause (.05)
   end
```

mixed oscillation. $\left(c = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/2 \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$

(setup v1, v2, v3, center...)

```
>> hold on
>> for t = 0:.1:6*pi
     pos = 1/2 * v1 * cos(t) + ...
           1/3 * v2 * cos(2*t) + ...
           1/2 * v3 * cos(3*t) + ...
           center
     scatter([t; t; t; t; t], [0; pos; -8])
     pause (.05)
   end
```